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Scattered Date Interpolation from Principal Shift-Invariant Spaces

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Under certain assumptions on the compactly supported function $\phi \in C(\mathbb{R}^d)$, we propose two methods of selecting a function s from the scaled principal shift-invariant space $S^h(\phi)$ such that s interpolates a given function f at a scattered set of data locations. For both methods, the selection scheme amounts to solving a quadratic programming problem and we are able to prove error estimates similar to those obtained by Duchon for surface spline interpolation.

1. INTRODUCTION

The scattered data interpolation problem in $\mathbb{R}^d$ is the following: Given a set of scattered points $\Xi \subset \mathbb{R}^d$ and a complex-valued function f defined at least on Ξ, one seeks a "nice" function s: $\mathbb{R}^d \to \mathbb{C}$ which interpolates the data $f_{|\Xi}$; that is, which satisfies $s(\xi) = f(\xi)\ \forall \xi \in \Xi$. The reader is referred to the surveys [8, 10–12] for descriptions of a variety of interpolation methods. One such method is that of surface spline interpolation (see [9]) which we now describe.

Let $m \in \mathbb{N} := \{1, 2, 3, \ldots\}$ be such that $m > d/2$, and let H^m denote the set of all tempered distributions f for which $D^\alpha f \in L_2 := L_2(\mathbb{R}^d)$ for all $|\alpha| = m$. For measurable $A \subset \mathbb{R}^d$ and $f \in H^m$, we define the seminorm

$$|f|_{H^m(A)} := (2\pi)^{d/2} \sqrt{\sum_{|\alpha| = m} \tau_\alpha \, \|D^\alpha f\|^2_{L_2(A)}}\,,$$

where the τ_α's are the positive integers determined by the equation $|x|^{2m} = \sum_{|\alpha|=m} \tau_\alpha x^{2\alpha}$, $x \in \mathbb{R}^d$. In case $A = \mathbb{R}^d$, we write simply $|f|_{H^m}$. The surface spline interpolation method dictates that $s \in H^m$ be chosen to minimize $|s|_{H^m}$ subject to the interpolation conditions $s_{|\Xi} = f_{|\Xi}$. If Ξ is finite and not

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contained in the zero-set of any nontrivial polynomial in $\Pi_{m-1} :=$ {polynomials of degree $\leqslant m-1$}, then the surface spline interpolant s can be realized as the unique function which interpolates the data $f_{|\Xi}$ and has the form $s = q + \sum_{\xi \in \Xi} \lambda_\xi \zeta(\cdot - \xi)$, where $q \in \Pi_{m-1}$, the λ_ξ's satisfy $\sum_{\xi \in \Xi} \lambda_\xi r(\xi) = 0 \ \forall r \in \Pi_{m-1}$, and ζ is the radially symmetric function

$$\zeta(x) = \begin{cases} |x|^{2m-d} & \text{if } d \text{ is odd}, \\ |x|^{2m-d} \log |x| & \text{if } d \text{ is even}, \end{cases} \qquad x \in \mathbb{R}^d.$$

In order to discuss the error between f and s, let us assume that Ω is open, bounded, and has the cone property, and assume also that $\Xi \subset \bar{\Omega} :=$ closure(Ω). The "fill distance" from Ξ to Ω is the quantity $\delta := \delta(\Xi, \Omega) := \sup_{x \in \Omega} \inf_{\xi \in \Xi} |x - \xi|$. Duchon [9] has shown that if s is the surface spline interpolant to f at Ξ, then

$$\|f - s\|_{L_p(\Omega)} \leqslant const\ \delta^{m-d/2+d/p} |f|_{H^m} \qquad \forall f \in H^m \tag{1.1}$$

for $2 \leqslant p \leqslant \infty$ and δ sufficiently small (see [18] for some interpolation methods with more general error estimates). What is interesting about the proof of (1.1) is that it hinges not on the fact that s minimizes $|s|_{H^m}$, but rather on the fact that $|s|_{H^m}$ is bounded by $const\ |f|_{H^m}$. The point being that the form of s is irrelevant. To obtain (1.1), all that is needed is that s interpolate $f_{|\Xi}$ while maintaining $|s|_{H^m} \leqslant const\ |f|_{H^m}$. With this in mind we consider interpolation from principal shift-invariant spaces.

Let ϕ: $\mathbb{R}^d \to \mathbb{C}$ be continuous and compactly supported. The semi-discrete convolution $\phi *' c$ between ϕ and a function c (defined at least on $\mathbb{Z}^d$) is defined by

$$\phi *' c := \sum_{j \in \mathbb{Z}^d} c(j)\, \phi(\cdot - j),$$

with convergence taken uniformly on compact sets. For $A \subset \mathbb{R}^d$ let

$$S(\phi, A) := \{\phi *' c : c(j) = 0 \text{ whenever } supp\ \phi(\cdot - j) \cap A = \varnothing\}.$$

The space $S(\phi, \mathbb{R}^d)$ is a *shift-invariant* space because $s(\cdot - j) \in S(\phi, \mathbb{R}^d)$ whenever $s \in S(\phi, \mathbb{R}^d)$ and $j \in \mathbb{Z}^d$. It is called a *principal* shift-invariant space because it is generated by the single function ϕ. For entry points into the vast literature on approximation from shift-invariant spaces, the reader is referred to [3, 4, 7]. The space $S(\phi, A)$ is refined by dilation for which we employ the dilation operator σ_h defined by

$$\sigma_h f := f(h\cdot).$$

For $h>0$ and $A\subset\mathbb{R}^d$ let

$$S^h(\phi, A) := \{\sigma_{1/h}s : s\in S(\phi, h^{-1}A)\}.$$

In other words, $S^h(\phi, A)$ is the closure, in the topology of uniform convergence on compact sets, of $span\{\phi(\cdot/h-j) : j\in\mathbb{Z}^d,\ supp\ \phi(\cdot/h-j)\cap A\neq\emptyset\}$. The approximation order of the scale of spaces $\{S^h(\phi, \mathbb{R}^d)\}_{h>0}$ can be characterized in terms of the Strang–Fix conditions:

DEFINITION 1.2. ϕ is said to satisfy the *Strang–Fix conditions of order m* ($m\in\mathbb{N}$) if $\hat{\phi}(0)\neq 0$ and $D^\alpha\hat{\phi}(2\pi j)=0\ \forall j\in\mathbb{Z}^d\backslash 0,\ |\alpha|<m$.

Here $\hat{\phi}$ denotes the Fourier transform of ϕ. It is known (see [6, 15]) that ϕ satisfies the Strang–Fix conditions of order m if and only if

$$\inf_{s\in S^h(\phi,\mathbb{R}^d)} \|f-s\|_{L_p} = O(h^m) \qquad \text{as} \qquad h\to 0 \qquad \forall f\in W_p^m, \qquad 1\leqslant p\leqslant\infty,$$

where W_p^m denotes the Sobolev space (see [2]) of all tempered distributions f for which $D^\alpha f\in L_p := L_p(\mathbb{R}^d)\ \forall\ |\alpha|\leqslant m$.

Let $\Omega\subset\mathbb{R}^d$ be open and bounded, and assume that ϕ satisfies the Strang–Fix conditions of order m for some $m\in\mathbb{N}$ with $m>d/2$. We show in Section 2 that if Ξ is a finite subset of $\bar{\Omega}$, then $S^h(\phi, \Omega)$ contains functions which interpolate $f_{|\Xi}$ whenever h is sufficiently small; precisely, whenever $0<h\leqslant sep(\Xi)/\varepsilon_\phi$, where ε_ϕ is a positive constant depending only on ϕ and where

$$sep(\Xi) := \inf\{|\xi-\xi'| : \xi, \xi'\in\Xi, \xi\neq\xi'\}$$

denotes the separation distance in Ξ. Of course, in this case, there are infinitely many functions in $S^h(\phi, \Omega)$ which interpolate $f_{|\Xi}$. In light of the discussion surrounding (1.1), a sensible way of selecting a particular interpolant $s\in S^h(\phi, \Omega)$ is to choose one which minimizes $|s|_{H^m(\Omega)}$. In Section 7, under the additional assumptions that $\phi\in W_2^m$ and that Ω is connected and has a Lipschitz boundary, we show that if s is chosen in this manner then (1.1) holds whenever δ is sufficiently small and $0<h\leqslant sep(\Xi)/\varepsilon_\phi$. The above method of selecting the particular interpolant from $S^h(\phi, \Omega)$ should be viewed as topological in nature; the topology being that of $H^m(\Omega)$. We mention that Shen and Waldron [21] have proposed an algebraic method of selecting an interpolant from $S^h(\phi, \Omega)$ when ϕ is a box spline. As with the current topological method, their algebraic method is motivated by a desire to select a smooth interpolant.

The additional assumption that $\phi\in W_2^m$ is very strong, and a quick survey of "distinguished" box-splines (see [5]) or B-splines reveals numerous examples where ϕ satisfies the Strang–Fix conditions of order m but $\phi\notin W_2^m$. For example, if M_1 is the box-spline associated with the $d+1$

directions $\{e_1, e_2, ..., e_d, e_1+e_2+\cdots+e_d\}$ and $M_k := M_{k-1} * M_1$, then M_k satisfies the Strang–Fix conditions of order $2k$, but $M_k \in W_2^{2k-1} \backslash W_2^{2k}$. Here, $\{e_i\}$ denotes the natural basis in $\mathbb{R}^d$, and the convolution is defined (as usual) by $f * g(x) := \int_{\mathbb{R}^d} f(x-t)\, g(t)\, dt$. A great share of the effort in the present work is devoted to replacing this assumption with the weaker assumption that $\phi \in W_2^\kappa$ for some $\kappa \in \mathbb{N}$ satisfying $d/2 < \kappa \leqslant m$. Note that this supports the abovementioned example M_k when $d/2 < 2k-1 =: \kappa$ and $m := 2k$.

Unfortunately, the cost functional $|s|_{H^m(\Omega)}$ is no longer meaningful when $\kappa < m$ for the simple reason that the functions in $S^h(\phi, \Omega)$ are not assumed to lie in H^m. This is very similar to the situation encountered in [17]. There, the natural choice of the cost functional was $|s|_{H^{2m}}$ but the functions s under consideration were spanned by translates of the function ζ (defined above) which does not locally belong to H^{2m}. This difficulty was overcome in [17] by using a cost functional of the form $|\delta^{-d}\eta(\cdot/\delta) * s|_{H^{2m}}$ where η is a well chosen exponentially decaying function. We employ a similar cure. In Section 3 we show that there exists a compactly supported distribution η such that $\hat{\eta} \sim (1+|\cdot|^2)^{(\kappa-m)/2}$. The cost functional

$$|h^{-d}\eta(\cdot/h) * s|_{H^m} \tag{1.3}$$

is now well defined because $\eta * \phi \in W_2^m$ (by Proposition 3.4). In order to obtain something like (1.1) we have to slightly adjust our approach. We assume only that Ω is open, bounded and has the cone property, and we let Ω_0 be any open, bounded set which contains $\bar{\Omega}$. With $0 < h \leqslant sep(\Xi)/\varepsilon_\phi$, we choose $s \in S^h(\phi, \Omega_0)$ to minimize (1.3) subject to the interpolation conditions $s_{|\Xi} = f_{|\Xi}$. In Section 6 we show that if δ is sufficiently small, then

$$\|f-s\|_{L_p(\Omega)} \leqslant const\; \delta^{m-d/2+d/p}\, \|f\|_{W_2^m} \qquad \forall 2 \leqslant p \leqslant \infty,\ f \in W_2^m,$$

where

$$\|f\|_{W_2^m} := \|(1+|\cdot|^2)^{m/2}\, \hat{f}\|_{L_2}.$$

An outline of the paper is as follows: In Section 2 we prove that interpolants from $S^h(\phi, \mathscr{A})$ exist whenever $0 < h \leqslant sep(\mathscr{A})/\varepsilon_\phi$, while in Section 3 we settle some technical issues related to the convolution $\eta * f$ when f is a tempered distribution. We show in Section 4 that the error is controlled by the cost functional (1.3). The operator norm of the operator $\phi *'$ is analyzed in various settings in Section 5. Finally, in Section 6 and Section 7, the two abovementioned interpolation schemes are described and analyzed.

Throughout this paper we use standard multi-index notation: $D^\alpha := (\partial^{\alpha_1}/\partial x_1^{\alpha_1})(\partial^{\alpha_2}/\partial x_2^{\alpha_2}) \cdots (\partial^{\alpha_d}/\partial x_d^{\alpha_d})$. For multi-indices α, we define $|\alpha| := \alpha_1+\alpha_2+\cdots+\alpha_d$, while for $x \in \mathbb{R}^d$, we define $|x| := \sqrt{x_1^2+x_2^2+\cdots+x_d^2}$.

The Fourier transform of a function f is defined formally by $\hat{f}(w) := \int_{\mathbb{R}^d} e^{-iw \cdot x} f(x)\, dx$ and plays an important role in the sequel. One related fact which follows from the Plancherel Theorem is that $|f|_{H^m}$ has the representation in the Fourier domain as $\| |\cdot|^m \hat{f} \|_{L_2(\mathbb{R}^d \setminus 0)}$ for all $f \in H^m$. It follows from this that $|\sigma_h f|_{H^m} = h^{m-d/2} |f|_{H^m}$ and $|f|_{H^m} \leqslant \|f\|_{W_2^m}$. The space of compactly supported C^∞ functions is denoted $C_c^\infty(\mathbb{R}^d)$. The space $C_c(A)$ is the set of all continuous functions having compact support contained in A. For a countable set $\mathscr{A}$, $\ell_p(\mathscr{A})$ denotes (as usual) the L_p space taken with respect to the counting measure in $\mathscr{A}$. We write simply ℓ_p for $\ell_p(\mathbb{Z}^d)$, and we use ℓ_0 to denote the space of finitely supported sequences defined on $\mathbb{Z}^d$. If μ is a distribution and g is a test function, then the application of μ to g is denoted $\langle g, \mu \rangle$. We employ the notation *const* to denote a generic constant in the range $(0, \infty)$ whose value may change with each occurrence. In the statement of results we specify the dependencies of any *const* while in proofs we omit the dependencies for the sake of brevity. Two oft employed subsets of $\mathbb{R}^d$ are the open unit ball $B := \{x \in \mathbb{R}^d : |x| < 1\}$ and the unit cube $C := [-1/2, 1/2)^d$.

2. EXISTENCE OF INTERPOLANTS FROM $S^h(\phi, \mathscr{A})$

The following lemma gives sufficient conditions for the existence of interpolants to f from $S^h(\phi, \mathscr{A})$, where $\mathscr{A}$ denotes a (possibly unbounded) subset of $\mathbb{R}^d$ satisfying $sep(\mathscr{A}) > 0$.

LEMMA 2.1. *Let $\phi \in C_c(\mathbb{R}^d)$ satisfy the Strang–Fix conditions of order $m \geqslant 1$. There exists $\varepsilon_\phi > 0$ (depending only on ϕ) such that if $0 < h \leqslant sep(\mathscr{A})/\varepsilon_\phi$* and $f \in \ell_2(\mathscr{A})$, then there exists $s \in S^h(\phi, \mathscr{A})$, say $s = \sigma_{1/h}(\phi *' c)$, such that $s_{|\mathscr{A}} = f_{|\mathscr{A}}$ and $\|c\|_{\ell_2} \leqslant const(\phi) \|f\|_{\ell_2(\mathscr{A})}$.

Proof. It suffices to consider the case $h = 1$ since the general case can then be obtained by scaling. It is known [16] that $\phi *' 1 = \hat{\phi}(0)$. Put $\mathscr{N} := \{j \in \mathbb{Z}^d : supp\, \phi(\cdot - j) \cap C \neq \varnothing\}$. Let $b\colon \mathbb{Z}^d \to \mathbb{C}$ be given by $b := \chi_{\mathscr{N}} / \hat{\phi}(0)$, and put $\psi := \phi *' b$. Note that $\psi = 1$ on C. Put $r := \max\{|x| : x \in \mathscr{N} \cup supp\, \psi\}$ and $\varepsilon_\phi := 2r + \sqrt{d}$. Assume $sep(\mathscr{A}) \geqslant \varepsilon_\phi$. For $x \in \mathbb{R}^d$, let $[x] \in \mathbb{Z}^d$ be defined by $x \in [x] + C$. Put $\tilde{c} := \sum_{a \in \mathscr{A}} f(a)\, b(\cdot - [a])$ and $\tilde{s} := \phi *' \tilde{c}$. The choice of ε_ϕ ensures that the supports of the sequences $\{b(\cdot - [a])\}_{a \in \mathscr{A}}$ are pairwise disjoint. Consequently, $\|\tilde{c}\|_{\ell_2}^2 = \sum_{a \in \mathscr{A}} |f(a)|^2 \|b(\cdot - [a])\|_{\ell_2}^2 = \|b\|_{\ell_2}^2 \|f\|_{\ell_2(\mathscr{A})}^2$. The choice of ε_ϕ also ensures that the supports of the functions $\{\psi(\cdot - [a])\}_{a \in \mathscr{A}}$ are pairwise disjoint. Hence, if $a \in \mathscr{A}$, then $\tilde{s}(a) = \sum_{a' \in \mathscr{A}} f(a')\, \psi(a - [a']) = f(a)\, \psi(a - [a]) = f(a)$. It may be the case that $\tilde{s} \notin S(\phi, \mathscr{A})$, so define $c\colon \mathbb{Z}^d \to \mathbb{C}$ by $c(j) := \tilde{c}(j)$, if $supp\, \phi(\cdot - j) \cap \mathscr{A} \neq \varnothing$, and $c(j)=0$ otherwise. Put $s:=\phi *' c$. Then $s(a)=\tilde{s}(a)=f(a)$ for all $a\in\mathscr{A}$ and $\|c\|_{\ell_2}\leqslant\|\tilde{c}\|_{\ell_2}=const\,\|f\|_{\ell_2}$. ▮

3. CONVOLUTION WITH THE DISTRIBUTION η

In this section we settle some technical issues related to our cost functional (1.3). We begin by proving the existence of the compactly supported distribution η mentioned in the introduction.

LEMMA 3.1. *Let $\kappa, m\in\mathbb{N}$ be such that $d/2<\kappa\leqslant m$. There exists a compactly supported distribution η such that*

(3.2)
$$const(m,d)(1+|w|^2)^{(\kappa-m)/2}\leqslant\hat{\eta}(w)\leqslant const(m,d)(1+|w|^2)^{(\kappa-m)/2}\qquad\forall w\in\mathbb{R}^d.$$

Proof. If $\kappa=m$, then we can simply take η to be the Dirac δ-distribution defined by $\langle g,\eta\rangle=g(0)$. So assume $d/2<\kappa<m$. Define $\eta_1\in L_1(\mathbb{R}^d)$ by

$$\eta_1(x):=|x|^{(m-\kappa-d)/2}\,K_{(m-\kappa-d)/2}(|x|),\qquad x\in\mathbb{R}^d,$$

where K_ν denotes the modified Bessel function of order ν (see [1]). It is known [13] that $\hat{\eta}_1=c(1+|\cdot|^2)^{(\kappa-m)/2}$ for some positive constant c (depending only on d and $m-\kappa$). Let $\zeta\in C_c(\mathbb{R}^d)$ satisfy $\zeta(0)=1$ and

$$0\leqslant\hat{\zeta}(w)\leqslant(1+|w|)^{-(d+m-\kappa+1)},\qquad w\in\mathbb{R}^d.$$

Such functions ζ can be easily realized as box-splines or tensor-product B-splines (see [5]). Define $\eta\in L_1(\mathbb{R}^d)$ by

$$\eta:=\zeta\eta_1.$$

The decay assumptions on $\hat{\eta}_1$ and $\hat{\zeta}$ ensure that $\hat{\eta}_1*\hat{\zeta}$ is well defined and that (3.2) holds with $\hat{\eta}_1*\hat{\zeta}(w)$ in place of $\hat{\eta}(w)$. So, in order to complete the proof, it suffices to show that

$$\hat{\eta}=(2\pi)^{-d}\,\hat{\eta}_1*\hat{\zeta}. \tag{3.3}$$

Note that (3.3) would be immediate if $\zeta\in C_c^\infty(\mathbb{R}^d)$. To establish (3.3) assuming only that $\zeta\in C_c(\mathbb{R}^d)$, we define $\zeta_n:=\sigma_n*\zeta$, where $\{\sigma_n\}$ is an approximate identity (i.e., $\sigma\in C_c^\infty(\mathbb{R}^d)$, $\int_{\mathbb{R}^d}\sigma=1$, $\sigma_n:=n^d\sigma(n\cdot)$). Then $\zeta_n\in C_c^\infty(\mathbb{R}^d)$ and

$$(\zeta_n\eta_1)^\wedge=(2\pi)^{-d}\,\hat{\eta}_1*\hat{\zeta}_n=(2\pi)^{-d}\,\hat{\eta}_1*(\hat{\sigma}_n\hat{\zeta}).$$

It is now a simple matter to obtain (3.3) by showing that (as $n \to \infty$) $\zeta_n \eta_1 \to \zeta \eta_1$ in $L_1(\mathbb{R}^d)$ and that $(\zeta_n \eta_1)^\wedge \to (2\pi)^{-d}\, \hat{\eta}_1 * \hat{\zeta}$ in $L_\infty(\mathbb{R}^d)$. ∎

With the existence of η settled, we turn now to the issue of defining the convolution $\eta * f$ assuming only that f is a tempered distribution. Our definition is valid not just for η, but for any tempered distribution whose Fourier transform lies in the space $\mathscr{M}$ defined below.

Let $\mathscr{S}$ denote the "rapidly decreasing functions" (see [20]) topologized (as usual) by the seminorms $\{\rho_n\}_{n \in \mathbb{N}}$, where

$$\rho_n(g) := \max_{|\alpha| \leqslant n} \|(1+|\cdot|)^n\, D^\alpha g\|_{L_\infty}.$$

Let $\mathscr{M}$ denote the set of all $g \in C^\infty(\mathbb{R}^d)$ which satisfy

$$\forall N \in \mathbb{N} \qquad \exists n \in \mathbb{N} \qquad \max_{|\alpha| \leqslant N} \|(1+|\cdot|)^{-n}\, D^\alpha g\|_{L_\infty} < \infty.$$

For example, if u is a compactly supported distribution, then it follows from a theorem of Paley–Wiener that $\hat{u} \in \mathscr{M}$. If $g \in \mathscr{M}$, then it is a consequence of Leibniz' formula that $g\zeta \in \mathscr{S}\ \forall \zeta \in \mathscr{S}$, and it is a consequence of the closed graph theorem that the mapping $\zeta \mapsto g\zeta$ is a continuous operator on $\mathscr{S}$. Consequently, the mapping $f \mapsto gf$ is a continuous operator on $\mathscr{S}'$ (the space of tempered distributions) whenever $g \in \mathscr{M}$.

Definition. Let u and v be tempered distributions with $\hat{u} \in \mathscr{M}$ or $\hat{v} \in \mathscr{M}$. The convolution $u * v$ is defined as the inverse Fourier transform of the tempered distribution $\hat{u}\hat{v}$:

$$u * v := (\hat{u}\hat{v})^\vee.$$

If $\hat{u} \in \mathscr{M}$, then it follows that $u*$ is a continuous operator on $\mathscr{S}'$. We collect in the following proposition several properties of the convolution operator $\eta*$ which will be used in the sequel.

Proposition 3.4. *Let κ, m, η be as in Lemma 3.1, and let $\phi \in W_2^\kappa$ be compactly supported. Put $\psi := \eta * \phi$. Then $\psi \in W_2^m$ and $\operatorname{supp} \psi \subset \operatorname{supp} \eta + \operatorname{supp} \phi$.* Let c: *$\mathbb{Z}^d \to \mathbb{C}$ have at most polynomial growth and for $n \in \mathbb{N}$ define $c_n \in \ell_0$ by*

$$c_n(j) := \begin{cases} c(j) & \text{if } |j| \leqslant n, \\ 0 & \text{else} \end{cases}.$$

Then

$$\text{(i)} \qquad \phi *' c_n \to \phi *' c \text{ in } \mathscr{S}' \qquad \text{and}$$

$$\text{(ii)} \qquad \eta * (\phi *' c) = \psi *' c.$$

Proof. To see that $\psi \in W_2^m$ note that by (3.2)

$$\|\psi\|_{W_2^m} = \|(1+|\cdot|^2)^{m/2}\, \hat{\eta}\hat{\phi}\|_{L_2} \leqslant const\, \|(1+|\cdot|^2)^{\kappa/2}\, \hat{\phi}\|_{L_2} = const\, \|\phi\|_{W_2^\kappa} < \infty.$$

That $supp\ \psi \subset supp\ \eta + supp\ \phi$ is proved in [14, Th. 4.9 and p. 87]. Let r be the smallest positive real number for which $supp\ \phi \subset r\bar{B}$. There exists a polynomial q, say of degree k, such that $|c(j)| \leqslant q(j)\ \forall j \in \mathbb{Z}^d$. If $g \in \mathscr{S}$, then

$$\begin{aligned} |\langle g, \phi *' c\rangle - \langle g, \phi *' c_n\rangle| &= |\langle g, \phi *' (c-c_n)\rangle| \leqslant \sum_{|j|>n} |c(j)|\, |\langle g, \phi(\cdot - j)\rangle| \\ &\leqslant \sum_{|j|>n} q(j)\, \|\phi\|_{L_1}\, \|g\|_{L_\infty(j+rB)} \\ &\leqslant const\, \Big(\sum_{|j|>n} q(j)(1+|j|)^{-k-d-1}\Big)\, \rho_{k+d+1}(g). \end{aligned}$$

Since $\sum_{|j|>n} q(j)(1+|j|)^{-k-d-1} \to 0$ as $n \to \infty$, we obtain (i). Since $\psi \in W_2^m$ has compact support, we have by (i) that $\psi *' c_n \to \psi *' c$ in $\mathscr{S}'$. Since $\eta *$ is a continuous operator on $\mathscr{S}'$, it follows from (i) that $\eta * (\phi *' c_n) \to \eta * (\phi *' c)$ in $\mathscr{S}'$. Noting that $\eta * (\phi *' c_n) = \psi *' c_n\ \forall n \in \mathbb{N}$, we obtain (ii). ∎

4. AN ERROR ESTIMATE

The following theorem contains our basic error estimate. In practice, the function g will be the error $f - s$. Of course, if s interpolates f at Ξ, then $f - s$ will vanish on Ξ.

THEOREM 4.1. *Let κ, m, η be as in Lemma* 3.1. *Let Ω be an open, bounded subset of $\mathbb{R}^d$ having the cone property and let $\Xi \subset \bar{\Omega}$. There exists $\delta_0 > 0$ such that if $\delta := \delta(\Xi; \Omega) \leqslant \delta_0$, then for $2 \leqslant p \leqslant \infty$*

$$\|g\|_{L_p(\Omega)} \leqslant const(\eta, m, \Omega)\, \delta^{m-d/2+d/p}\, |\delta^{-d}\eta(\cdot/\delta) * g|_{H^m}$$
$$\forall g \in H^m + H^\kappa \text{ satisfying } g_{|\Xi} = 0.$$

We mention that in the case $\kappa = m$, the above conclusion reduces to

$$\|g\|_{L_p(\Omega)} \leqslant const(m, \Omega)\, \delta^{m-d/2+d/p}\, |g|_{H^m} \qquad \forall g \in H^m \text{ which vanish on } \Xi,$$

which is known [9]. Our proof of this theorem requires two supporting lemmas. The proof of the first is essentially the same as the proof of

[17, Prop. 3.1] if one replaces $|f|_{H^m}$ with $|f|_{H^\kappa}$, $|f|_{H^{2m}}$ with $|f|_{H^m}$, and $|f|_*$ with $|\eta * f|_{H^m}$.

LEMMA 4.2. *Let κ, m, η be as in Lemma 3.1, and let $r>0$. For each $j \in \mathbb{Z}^d$, let $\mathcal{N}_j$ be a finite subset of $j+rB$. If $\{b_{j,\xi}\}_{j \in \mathbb{Z}^d, \xi \in \mathcal{N}_j}$ is such that*

$$\sum_{\xi \in \mathcal{N}_j} b_{j,\xi} q(\xi) = 0 \quad \forall q \in \Pi_{m-1}, j \in \mathbb{Z}^d \quad \text{and} \quad M := \sup_{j \in \mathbb{Z}^d} \sum_{\xi \in \mathcal{N}_j} |b_{j,\xi}| < \infty,$$

then

$$\sum_{j \in \mathbb{Z}^d} \left| \sum_{\xi \in \mathcal{N}_j} b_{j,\xi} f(\xi) \right|^2 \leqslant \text{const}(\eta, m, r)\, M^2\, |\eta * f|^2_{H^m} \qquad \forall f \in H^m + H^\kappa.$$

The following lemma is taken from [17, Lemma 4.1].

LEMMA 4.3. *Let $n \geqslant 0$. If $\Omega \subset \mathbb{R}^d$ is bounded, open, and has the cone property, then there exists $\delta_0, r_0 \in (0, \infty)$ (depending only on n and Ω) such that if Ξ is a finite subset of $\bar{\Omega}$ with $\delta := \delta(\Xi; \Omega) \leqslant \delta_0$, then for all $x \in \Omega/\delta$ there exists a finite $\mathcal{N} \subset (\Xi/\delta) \cap (x + r_0 B)$ and $\{b_\xi\}_{\xi \in \mathcal{N}}$ such that*

$$q(x) + \sum_{\xi \in \mathcal{N}} b_\xi q(\xi) = 0 \qquad \forall q \in \Pi_n \qquad \text{and} \qquad \sum_{\xi \in \mathcal{N}} |b_\xi| \leqslant \text{const}(n, \Omega).$$

Proof of Theorem 4.1. Let δ_0, r_0 be as in Lemma 4.3 with $n = m-1$. Put $\mathcal{A} := \{j \in \mathbb{Z}^d : (j+C) \cap (\Omega/\delta) \neq \emptyset\}$. For each $j \in \mathcal{A}$, let $x_j \in (j+C) \cap (\Omega/\delta)$ be such that $\|\sigma_\delta g\|_{L_\infty((j+C)\cap(\Omega/\delta))} \leqslant 2\,|g(\delta x_j)|$. By Lemma 4.3, for each $j \in \mathcal{A}$, there exists $\mathcal{N}_j \subset (\Xi/\delta) \cap (x_j + r_0 B)$ and $\{b_{j,\xi}\}_{\xi \in \mathcal{N}_j}$ such that

$$q(x_j) + \sum_{\xi \in \mathcal{N}_j} b_{j,\xi} q(\xi) = 0 \qquad \forall q \in \Pi_{m-1} \qquad \text{and} \qquad \sum_{\xi \in \mathcal{N}_j} |b_{j,\xi}| \leqslant \text{const}(m, \Omega).$$

Put $r := r_0 + \sqrt{d}/2$ and note that $\{x_j\} \cup \mathcal{N}_j \subset j + rB$ for all $j \in \mathcal{A}$. Now,

$$
\begin{aligned}
\|g\|_{L_p(\Omega)} &= \delta^{d/p}\, \|\sigma_\delta g\|_{L_p(\Omega/\delta)} \\
&\leqslant \delta^{d/p}\, \|j \mapsto \|\sigma_\delta g\|_{L_\infty((j+C)\cap(\Omega/\delta))}\|_{\ell_p(\mathcal{A})} \\
&\leqslant 2\delta^{d/p}\, \|j \mapsto g(\delta x_j)\|_{\ell_p(\mathcal{A})} \\
&\leqslant 2\delta^{d/p}\, \|j \mapsto g(\delta x_j)\|_{\ell_2(\mathcal{A})}, \qquad \text{since} \quad 2 \leqslant p, \\
&= 2\delta^{d/p} \sqrt{\sum_{j \in \mathcal{A}} |g(\delta x_j)|^2}\,.
\end{aligned} \tag{4.4}
$$

Since $g(\delta\xi)=0$ for all $\xi \in \Xi/\delta$, we have

$$|g(\delta x_j)| = \left| g(\delta x_j) + \sum_{\xi \in \mathcal{N}_j} b_{j,\xi}\, g(\delta\xi) \right|, \qquad \forall j \in \mathscr{A}.$$

We thus obtain from (4.4) and Lemma 4.2 that

$$\|g\|_{L_p(\Omega)} \leqslant const\ \delta^{d/p}\, |\eta * (\sigma_\delta g)|_{H^m} = const\ \delta^{m-d/2+d/p}\, |\delta^{-d}\eta(\cdot/\delta) * g|_{H^m}. \quad \blacksquare$$

5. AN ANALYSIS OF $\phi *'$

As mentioned just prior to the statement of Theorem 4.1, our error estimates will employ Theorem 4.1 with $g = f - s$. Roughly speaking, the factor $|\delta^{-d}\eta(\cdot/\delta) * (f-s)|_{H^m}$ will be estimated by $|\delta^{-d}\eta(\cdot/\delta) * f|_{H^m} + |\delta^{-d}\eta(\cdot/\delta) * s|_{H^m}$, where the first term will be shown to be bounded by a constant times $|f|_{H^m}$. The second term is our cost functional (1.3) with δ in place of h. Although this second term involves the parameter δ, its action on any $s \in S^\delta(\phi, \mathbb{R}^d)$ exhibits a certain stationarity. Namely, if $s = \sigma_{1/\delta}(\phi *' c)$, then

$$\delta^{m-d/2}\, |\delta^{-d}\eta(\cdot/\delta) * s|_{H^m} = |\eta * (\phi *' c)|_{H^m}. \tag{5.1}$$

Thus, the right side of (5.1) is an important quantity. Two estimates of this quantity are given in the following proposition.

PROPOSITION 5.2. *Let κ, m, η be as in Lemma* 3.1, *and let $\phi \in W_2^\kappa$ be compactly supported. Then*

$$|\eta * (\phi *' f)|_{H^m} \leqslant const(\eta, m, \phi)\, \|f\|_{\ell_2} \qquad \forall f \in \ell_2. \tag{5.3}$$

If, in addition, ϕ satisfies the Strang–Fix conditions of order m, then

$$|\eta * (\phi *' f)|_{H^m} \leqslant const(\eta, m, \phi)\, |f|_{H^m} \qquad \forall f \in H^m. \tag{5.4}$$

Our proof of this proposition requires the following lemma which is a consequence of [19, Théorème 1.6] and the Sobolev embedding theorem [2, p. 97].

LEMMA 5.5. *Let $y \in \mathbb{R}^d$, $r > 0$, and $m \in \mathbb{N}$ with $m > d/2$. For all $f \in H^m$ there exists $q \in \Pi_{m-1}$ such that*

$$\|f - q\|_{L_\infty(y+rB)} \leqslant const(d, m, r)\, |f|_{H^m(y+rB)}.$$

Proof of Proposition 5.2. Put $\psi := \eta * \phi$. By Proposition 3.4, $\psi \in W_2^m$ is compactly supported and

$$\eta * (\phi *' f) = \psi *' f. \tag{5.6}$$

Put $\mathcal{N} := \{j \in \mathbb{Z}^d : supp\, \psi(\cdot - j) \cap C \neq \emptyset\}$, and note that $\# \mathcal{N} < \infty$. In consideration of (5.3), assume $f \in \ell_2$. Then

$$|\psi *' f|_{H^m}^2 = \sum_{\ell \in \mathbb{Z}^d} |\psi *' f|_{H^m(\ell + C)}^2 = \sum_{\ell \in \mathbb{Z}^d} \left| \sum_{j \in \mathcal{N}} f(\ell + j)\, \psi(\cdot - j) \right|_{H^m(C)}^2$$

$$\leqslant const \sum_{\ell \in \mathbb{Z}^d} |\psi|_{H^m}^2 \sum_{j \in \mathcal{N}} |f(\ell + j)|^2 \leqslant const\, \|f\|_{\ell_2}^2$$

which, in view of (5.6), proves (5.3). In consideration of (5.4), assume ϕ satisfies the Strang–Fix conditions of order m, and let $f \in H^m$. Since $\hat{\psi} = \hat{\eta}\hat{\phi}$, it follows that ψ also satisfies the Strang–Fix conditions of order m. Consequently, $\psi *' q \in \Pi_{m-1}$ for all $q \in \Pi_{m-1}$ (see [16]). Let r be the smallest number in $[\sqrt{d}/2, \infty)$ satisfying $\mathcal{N} \subset r\bar{B}$. By Lemma 5.5, for each $\ell \in \mathbb{Z}^d$ there exists $q_\ell \in \Pi_{m-1}$ such that

$$\|f - q_\ell\|_{L_\infty(\ell + rB)} \leqslant const\, |f|_{H^m(\ell + rB)}.$$

This yields the estimate

$$\begin{aligned} |\psi *' f|_{H^m(\ell + C)} &= |\psi *' (f - q_\ell)|_{H^m(\ell + C)} \\ &= \left| \sum_{j \in \mathcal{N}} (f(\ell + j) - q_\ell(\ell + j))\, \psi(\cdot - j) \right|_{H^m(\ell + C)} \\ &\leqslant \# \mathcal{N}\, \|f - q_\ell\|_{L_\infty(\ell + rB)}\, |\psi|_{H^m} \leqslant const\, |f|_{H^m(\ell + rB)}. \end{aligned}$$

Therefore,

$$|\psi *' f|_{H^m}^2 = \sum_{\ell \in \mathbb{Z}^d} |\psi *' f|_{H^m(\ell + C)}^2 \leqslant const \sum_{\ell \in \mathbb{Z}^d} |f|_{H^m(\ell + rB)}^2 \leqslant const\, |f|_{H^m}^2$$

which, in view of (5.6), proves (5.4). ∎

Our proof of the following result uses the standard quasi-interpolation argument (see [5, Chap. III]) which greatly simplifies when $m > d/2$.

PROPOSITION 5.7. *Let $\psi \in C_c(\mathbb{R}^d)$ and $m \in \mathbb{N}$ with $m > d/2$ be such that $\psi *' q = q\ \forall q \in \Pi_{m-1}$. If $sep(\mathscr{A}) \geqslant const$*, then

$$\|f - \psi *' f\|_{\ell_2(\mathscr{A})} \leqslant const(m, \psi)\, |f|_{H^m} \qquad \forall f \in H^m.$$

Proof. Let $\mathcal{N}$, r, and $\{q_\ell\}_{\ell \in \mathbb{Z}^d}$ be as defined in the proof of Proposition 5.2. Then for $\ell \in \mathbb{Z}^d$

$$\begin{aligned}\|f-\psi *' f\|_{L_\infty(\ell+C)} &= \|f-q_\ell-\psi *' (f-q_\ell)\|_{L_\infty(\ell+C)} \\ &\leqslant \|f-q_\ell\|_{L_\infty(\ell+C)} + \left\| \sum_{j \in \mathcal{N}} (f(\ell+j)-q_\ell(\ell+j))\, \psi(\cdot - j) \right\|_{L_\infty(C)} \\ &\leqslant (1+\#\mathcal{N}\ \|\psi\|_{L_\infty})\, \|f-q_\ell\|_{L_\infty(\ell+rB)} \leqslant const\, |f|_{H^m(\ell+rB)}.\end{aligned}$$

Since $sep(\mathscr{A}) \geqslant const$, we have

$$\begin{aligned}\|f-\psi *' f\|^2_{\ell_2(\mathscr{A})} &\leqslant const \sum_{\ell \in \mathbb{Z}^d} \|f-\psi *' f\|^2_{L_\infty(\ell+C)} \\ &\leqslant const \sum_{\ell \in \mathbb{Z}^d} |f|^2_{H^m(\ell+rB)} \leqslant const\, |f|^2_{H^m}. \quad \blacksquare\end{aligned}$$

6. AN INTERPOLATION METHOD FOR THE CASE $\kappa \leqslant m$

In the following, the phrase *nearly minimize* means to bring to within a constant factor of the minimal value. For example, to choose $g \in G$ to nearly minimize $\|g\|$ means to choose $g \in G$ so that $\|g\| \leqslant const \inf\{\|\tilde{g}\| : \tilde{g} \in G\}$.

Interpolation method 6.1. Let κ, m, η be as in Lemma 3.1. Let $\phi \in W_2^\kappa$ be compactly supported and satisfy the Strang–Fix conditions of order m, and let ε_ϕ be as in Lemma 2.1. Let Ω be an open, bounded subset of $\mathbb{R}^d$ having the cone property, and let Ω_0 be an open, bounded set which contains $\bar{\Omega}$. Let Ξ be a finite subset of $\bar{\Omega}$ and let $0 < h \leqslant sep(\Xi)/\varepsilon_\phi$. Choose $s \in S^h(\phi, \Omega_0)$ to nearly minimize $|h^{-d}\eta(\cdot/h) * s|_{H^m}$ subject to the interpolation conditions $s_{|\Xi} = f_{|\Xi}$. There exists $\delta_1 > 0$ such that if $\delta := \delta(\Xi, \Omega) \leqslant \delta_1$, then for all $f \in W_2^m$

(i) $|h^{-d}\eta(\cdot/h) * s|_{H^m} \leqslant const(\eta, m, \Omega, \Omega_0, \phi)\, \|f\|_{W_2^m}$ and

(ii) $\|f-s\|_{L_p(\Omega)} \leqslant const(\eta, m, \Omega, \Omega_0, \phi)\, \delta^{m-d/2+d/p}\, \|f\|_{W_2^m} \quad \forall 2 \leqslant p \leqslant \infty.$

Remark 6.2. The interpolant s can be found by *nearly* solving a quadratic programming problem. To see this, let s be written as $s = \sum_{j=1}^M c_j \phi(\cdot/h - k_j)$, where $\{k_1, k_2, ..., k_M\} := \{k \in \mathbb{Z}^d : supp\, \phi(\cdot/h-k) \cap \Omega_0 \neq \varnothing\}$, and put $\{\xi_1, \xi_2, ..., \xi_N\} := \Xi$. The interpolation conditions become $Ac = F$ where A is the $N \times M$ matrix having (i, j)-entry $\phi(\xi_i/h - k_j)$ and $F = [f(\xi_i)]_{1 \leqslant i \leqslant N}$. Put $\psi := \eta * \phi$ and let G be the $M \times M$ matrix having (i, j)-entry

$(\psi, \psi(\cdot + k_j - k_i))_{H^m}$, where $(,)_{H^m}$ denotes the semi-inner product associated with $|\cdot|_{H^m}$. The cost functional can then be written as

$$|h^{-d}\eta(\cdot/h) * s|_{H^m} = h^{-m+d/2}\sqrt{c^* G c},$$

where c^* denotes the complex conjugate of the transpose of c. Thus c is any near solution of the quadratic programming problem

$$\text{minimize } c^* G c$$

$$\text{subject to } Ac = F.$$

We mention that the matrices A and G are sparse in the sense that the number of nonzero entries in each row or column is bounded independently of M and N.

Proof of 6.1. Let $\varepsilon > 0$ be the largest positive real number for which $\Omega + \varepsilon B \subset \Omega_0$, and let $\zeta \in C_c^\infty(\Omega + (\varepsilon/2) B)$ be such that $\zeta = 1$ on Ω. The assumptions on ϕ ensure (see [16]) that there exists a finitely supported sequence $a: \mathbb{Z}^d \to \mathbb{C}$ such that $\psi := \phi *' a$ satisfies the Strang–Fix conditions of order m and the condition $\psi *' q = q$ for all $q \in \Pi_{m-1}$. Let δ_0 be as in Theorem 4.1, and let $\delta_1 \in (0, \delta_0]$ be sufficiently small to ensure that

$$\delta(\Xi, \Omega) \leqslant \delta_1, 0 < h \leqslant sep(\Xi)/\varepsilon_\phi$$
$$\text{and} \qquad supp\, g \subset h^{-1}(\Omega + (\varepsilon/2) B) \Rightarrow \psi *' g \in S(\phi, h^{-1}\Omega_0).$$

Let $f \in W_2^m$ and put $\tilde{f} := \zeta f$. Then $\|\tilde{f}\|_{W_2^m} \leqslant const\, \|f\|_{W_2^m}$. Assume $\delta := \delta(\Xi, \Omega) \leqslant \delta_1$. Put $s_1 := \sigma_{1/h}(\psi *' \sigma_h \tilde{f}) \in S^h(\phi, \Omega_0)$. Since $0 < h \leqslant sep(\Xi)/\varepsilon_\phi$, it follows by Lemma 2.1 that there exists $s_2 \in S^h(\phi, \Xi)$, say $s_2 = \sigma_{1/h}(\phi *' c)$, such that $\|c\|_{\ell_2} \leqslant const\, \|\tilde{f} - s_1\|_{\ell_2(\Xi)}$ and $s_2(\xi) = \tilde{f}(\xi) - s_1(\xi)$ for all $\xi \in \Xi$. Put $\tilde{s} := s_1 + s_2 \in S^h(\phi, \Omega_0)$, and note that $\tilde{s}(\xi) = s_1(\xi) + \tilde{f}(\xi) - s_1(\xi) = f(\xi)$ for all $\xi \in \Xi$. Consequently,

$$\begin{aligned} |h^{-d}\eta(\cdot/h) * s|_{H^m} &\leqslant const\, |h^{-d}\eta(\cdot/h) * \tilde{s}\,|_{H^m} = const\, h^{-m+d/2}\, |\eta * \sigma_h \tilde{s}\,|_{H^m} \\ &\leqslant const\, h^{-m+d/2}(|\eta * \sigma_h s_1|_{H^m} + |\eta * \sigma_h s_2|_{H^m}). \end{aligned} \tag{6.2}$$

By Proposition 5.2, we have

$$|\eta * \sigma_h s_1|_{H^m} = |\eta * (\psi *' \sigma_h \tilde{f}\,)|_{H^m} \leqslant const\, |\sigma_h \tilde{f}|_{H^m}$$

and

$$\begin{aligned} |\eta * \sigma_h s_2|_{H^m} &= |\eta * (\phi *' c)|_{H^m} \leqslant const\, \|c\|_{\ell_2} \\ &\leqslant const\, \|\tilde{f} - s_1\|_{\ell_2(\Xi)} = const\, \|\sigma_h \tilde{f} - \psi *' \sigma_h \tilde{f}\|_{\ell_2(\Xi/h)} \leqslant const\, |\sigma_h \tilde{f}|_{H^m} \end{aligned}$$

by Proposition 5.7. Therefore, by (6.2),

$$|h^{-d}\eta(\cdot/h)*s|_{H^m} \leqslant const\, h^{-m+d/2}\, |\sigma_h \tilde{f}|_{H^m} = const\, |\tilde{f}|_{H^m}$$
$$\leqslant const\, \|\tilde{f}\|_{W_2^m} \leqslant const\, \|f\|_{W_2^m}$$

which proves (i). Since $h \leqslant const\, \delta$, it follows that

$$|\delta^{-d}\eta(\cdot/\delta)*s|_{H^m} \leqslant const\, |h^{-d}\eta(\cdot/h)*s|_{H^m}.$$

Hence, by Theorem 4.1,

$$\|f-s\|_{L_p(\Omega)} \leqslant const\, \delta^{m-d/2+d/p}\, |\delta^{-d}\eta(\cdot/\delta)*(f-s)|_{H^m}$$
$$\leqslant const\, \delta^{m-d/2+d/p}(|\delta^{-d}\eta(\cdot/\delta)*f|_{H^m} + |\delta^{-d}\eta(\cdot/\delta)*s|_{H^m})$$
$$\leqslant const\, \delta^{m-d/2+d/p}\, \|f\|_{W_2^m}$$

which proves (ii). ▮

7. AN INTERPOLATION METHOD FOR THE CASE WHEN $\phi \in W_2^m$

The conclusion of the following result is an improvement over that of 6.1 as $|f|_{H^m(\Omega)}$ has taken the place of $\|f\|_{W_2^m}$ in (i) and (ii). To obtain this improvement, we have assumed further that $\phi \in W_2^m$ and that Ω is connected and has a Lipschitz boundary.

Interpolation method 7.1. Let $m \in \mathbb{N}$ with $d/2 < m$, and let $\phi \in W_2^m$ be compactly supported and satisfy the Strang–Fix conditions of order m. Let Ω be an open, bounded, connected subset of $\mathbb{R}^d$ having the cone property and a Lipschitz boundary (in the sense of [19]), and let δ_0 and ε_ϕ be as in Theorem 4.1 and Lemma 2.1, respectively. Let Ξ be a finite subset of $\bar{\Omega}$ and let $0 < h \leqslant sep(\Xi)/\varepsilon_\phi$. Let Ω_h be any measurable set which contains Ω, and let $s \in S^h(\phi, \Omega_h)$ be chosen to nearly minimize $|s|_{H^m(\Omega_h)}$ subject to the interpolation conditions $s_{|\Xi} = f_{|\Xi}$. If $\delta := \delta(\Xi, \Omega) \leqslant \delta_0$, then for all $f \in H^m$

(i) $|s|_{H^m(\Omega_h)} \leqslant const(m, \Omega, \phi)\, |f|_{H^m(\Omega)}$ and

(ii) $\|f-s\|_{L_p(\Omega)} \leqslant const(m, \Omega, \phi)\, \delta^{m-d/2+d/p}\, |f|_{H^m(\Omega)}$ $\quad \forall 2 \leqslant p \leqslant \infty$.

Remark. The interpolant s can be found by *nearly* solving the same quadratic programming problem described in Remark 6.2 excepting that $\{k_1, k_2, ..., k_M\} := \{k \in \mathbb{Z}^d : supp\ \phi(\cdot/h-k) \cap \Omega_h \neq \varnothing\}$ and $G(i, j) := (\phi, \phi(\cdot+k_j-k_i))_{H^m(h^{-1}\Omega_h)}$. If Ω_h is a complicated set, then the computation of G will likely be difficult. One way to ease this task is to choose Ω_h as

$$\Omega_h := \bigcup_{\ell \in \mathscr{A}_h} h(\ell+C),$$

where $\mathscr{A}_h := \{\ell \in \mathbb{Z}^d : \Omega \cap h(\ell+C) \neq \varnothing\}$. Using the auxiliary function $u: \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{C}$ given by $u(k, \ell) := (\phi, \phi(\cdot-k))_{H^m(\ell+C)}$ (which has a fixed number of nonzero entries), we can compute $G(i, j)$ as

$$G(i, j) = \sum_{\ell \in \mathscr{A}_h} u(k_i - k_j, \ell).$$

Our proof of 7.1 requires the following result which comes out of [9, p. 331].

THEOREM 7.2. *Let $m \in \mathbb{N}$ with $m > d/2$. If Ω is an open, bounded, connected subset of $\mathbb{R}^d$ having the cone property and a Lipschitz boundary (in the sense of [19]), then for all $f \in H^m$ there exists $f_\Omega \in H^m$ such that*

(i) $f_\Omega = f$ *on* Ω *and*

(ii) $|f_\Omega|_{H^m} \leqslant const(m, \Omega)\, |f|_{H^m(\Omega)}$.

Proof of 7.1. Let a, ψ be as in the proof of 6.1. Let $f \in H^m$ and let f_Ω be as in Theorem 7.2. Put $s_1 := \sigma_{1/h}(\psi *' \sigma_h f_\Omega)$. By Proposition 5.2, $s_1 \in H^m$ and

$$|\sigma_h s_1|_{H^m} \leqslant const\ |\sigma_h f_\Omega|_{H^m}. \tag{7.3}$$

Since $0 < h \leqslant sep(\Xi)/\varepsilon_\phi$, it follows by Lemma 2.1 that there exists $s_2 \in S^h(\phi, \Xi)$, say $s_2 = \sigma_{1/h}(\phi *' c)$, such that $\|c\|_{\ell_2} \leqslant const\ \|f_\Omega - s_1\|_{\ell_2(\Xi)}$ and $s_2(\xi) = f_\Omega(\xi) - s_1(\xi)$ for all $\xi \in \Xi$. Put $s_3 := s_1 + s_2 \in S^h(\phi, \mathbb{R}^d)$. Then $s_{3|\Xi} = f_{\Omega|\Xi} = f_{|\Xi}$, and

$$\begin{aligned} |s_3|_{H^m} &\leqslant |s_1|_{H^m} + |s_2|_{H^m} = h^{-m+d/2}(|\sigma_h s_1|_{H^m} + |\sigma_h s_2|_{H^m}) \\ &= h^{-m+d/2}(|\psi *' \sigma_h f_\Omega|_{H^m} + |\phi *' c|_{H^m}) \leqslant const\ h^{-m+d/2}(|\sigma_h f_\Omega|_{H^m} + \|c\|_{\ell_2}) \end{aligned}$$

by Proposition 5.2. Since $\|c\|_{\ell_2} \leqslant const\ \|f_\Omega - s_1\|_{\ell_2(\Xi)} = const\ \|\sigma_h f_\Omega - \psi *' \sigma_h f_\Omega\|_{\ell_2(\Xi/h)}$, we have by Proposition 5.7, that $\|c\|_{\ell_2} \leqslant const\ |\sigma_h f_\Omega|_{H^m}$. Therefore,

$$|s_3|_{H^m} \leqslant const\ h^{-m+d/2}\ |\sigma_h f_\Omega|_{H^m} = const\ |f_\Omega|_{H^m} \leqslant const\ |f|_{H^m(\Omega)} \tag{7.4}$$

by Theorem 7.2. Let $s_4 \in S^h(\phi, \Omega_h)$ be such that $s_4 = s_3$ on Ω_h. Then $s_{4_{|\Xi}} = s_{3_{|\Xi}} = f_{|\Xi}$. Hence,

$$|s|_{H^m(\Omega_h)} \leqslant const\, |s_4|_{H^m(\Omega_h)} = const\, |s_3|_{H^m(\Omega_h)} \leqslant const\, |s_3|_{H^m} \leqslant const\, |f|_{H^m(\Omega)}$$

which proves (i). By Theorem 7.2, there exists $s_\Omega \in H^m$ such that $s_\Omega = s$ on Ω and $|s_\Omega|_{H^m} \leqslant const\, |s|_{H^m(\Omega)}$. Hence, by Theorem 4.1,

$$\begin{aligned} \|f - s\|_{L_p(\Omega)} &= \|f_\Omega - s_\Omega\|_{L_p(\Omega)} \leqslant const\, \delta^{m-d/2+d/p}\, |f_\Omega - s_\Omega|_{H^m} \\ &\leqslant const\, \delta^{m-d/2+d/p}(|f_\Omega|_{H^m} + |s_\Omega|_{H^m}) \leqslant const\, \delta^{m-d/2+d/p}\, |f|_{H^m(\Omega)} \end{aligned}$$

which proves (ii). ∎

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REFERENCES

1. M. Abramowitz and I. A. Stegun, "Handbook of Mathematical Functions," Dover, New York, 1970.
2. R. A. Adams, "Sobolev Spaces," Academic Press, New York, 1975.
3. C. de Boor, R. A. DeVore, and A. Ron, Approximation from shift-invariant subspaces of $L_2(\mathbb{R}^d)$, *Trans. Amer. Math. Soc.* **341** (1994), 787–806.
4. C. de Boor, R. A. DeVore, and A. Ron, Approximation orders of FSI Spaces in $L_2(\mathbb{R}^d)$, *Constr. Approx.* **14** (1998), 631–652.
5. C. de Boor, K. Höllig, and S. Riemenschneider, "Box Splines," Springer-Verlag, New York, 1993.
6. C. de Boor and R.-Q. Jia, Controlled approximation and a characterization of the local approximation order, *Proc. Amer. Math. Soc.* **95** (1985), 547–553.
7. C. de Boor and A. Ron, Fourier analysis of the approximation power of principal shift-invariant spaces, *Constr. Approx.* **8** (1992), 427–462.
8. M. D. Buhmann, New developments in the theory of radial basis function interpolation, *in* "Multivariate Approximation: From CAGD to Wavelets" (K. Jetter and F. I. Utreras, Eds.), pp. 35–75, World Scientific Singapore, 1993.
9. J. Duchon, Sur l'erreur d'interpolation des fonctions de plusieur variables par les D^m-splines, *RAIRO Anal. Numer.* **12** (1978), 325–334.
10. N. Dyn, Interpolation of scattered data by radial functions, *in* "Topics in Multivariate Approximation" (C. K. Chui, L. L. Schumaker, and F. I. Utreras, Eds.), pp. 47–61, Academic Press, San Diego, 1987.
11. N. Dyn, Interpolation and approximation by radial and related functions, *in* "Approximation Theory VI" (C. K. Chui, L. L. Schumaker, and J. D. Ward, Eds.), Vol. 1, pp. 211–234, Academic Press, San Diego, 1989.

12. T. A. Foley and H. Hagen, Advances in scattered data interpolation, *Surv. Math. Ind.* **4** (1994), 7–84.
13. I. M. Gelfand and G. E. Shilov, "Generalized Functions," Vol. 1, Academic Press, San Diego, 1964.
14. M. A. Al-Gwaiz, "Theory of Distributions," Marcel Dekker, New York/Basel/Hong Kong, 1992.
15. R.-Q. Jia, The Toeplitz theorem and its applications to approximation theory and linear PDE's, *Trans. Amer. Math. Soc.* **347** (1995), 2585–2594.
16. R.-Q. Jia and J. Lei, Approximation by multiinteger translates of functions having global support, *J. Approx. Theory* **72** (1993), 2–23.
17. M. J. Johnson, Overcoming the boundary effects in surface spline interpolation, *IMA J. Numer. Anal.* **20** (2000), 405–422.
18. O. V. Matveev, Methods of approximate reconstruction of functions defined on chaotic lattices, *Izvestiya: Math.* **60** (1996), 985–1025.
19. J. Nečas, "Les Methodés Directes en Théorie des Equations Elliptiques," Masson, Paris, 1967.
20. W. Rudin, "Functional Analysis," McGraw–Hill, New York, 1973.
21. Z. Shen and S. Waldron, Scattered data interpolation by box splines, manuscript.